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INTEGRATION OF COUPLED NONLINEAR EQUATIONS IN BOUNDARY-LAYER THEORY WITH SPECIFIC REFERENCE TO HEAT TRANSFER NEAR THE STAGNATION POINT IN THREE-DIMENSIONAL FLOW

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Abstract—Equations of motion and energy governing a general three-dimensional flow of an incompressible fluid near a stagnation point are integrated analytically. Local skin friction and heat transfer are determined when the surface is isothermal. Whenever possible, these results are compared with available numerical solutions and found to be highly accurate. They show that the asymptotic method of solution of the boundary-layer equations retains its accuracy when applied to a system of coupled nonlinear equations, proper care being taken in summing divergent series by Euler's method.

NOMENCLATURE

a, b,	constants related to x, y velocity components of irrotational flow, equation (2.9);
c,	geometric parameter, $c = b/a$;
f, g,	dimensionless velocity functions, equation (2.11);
h_i,	coefficients related to elements of length in an orthogonal system, i—1, 2, 3;
p,	static pressure;
P,	stagnation point on surface S, origin of coordinate system;
Pr,	Prandtl number;
S,	surface;
T,	temperature;
u, v, w,	x, y, z velocity components in the boundary layer;
U, V,	x, y velocity components of the main stream;
x, y, z,	local orthogonal coordinate system on S;
$z\zeta$,	$= (a/\nu)^{\frac{1}{2}}$, dimensionless coordinate in the direction of local normal.

Greek symbols

ξ, η, ζ,	local orthogonal coordinate system (different from x, y, z);
ν,	kinematic viscosity;
ρ,	density;
τ,	an independent variable, equation (2.29);
θ,	dimensionless temperature; equation (2.13).

Subscripts

w,	refers to surface values;
∞,	refers to values at edge of boundary layer.

I. INTRODUCTION

MEKSYN has shown in a series of papers that the asymptotic method of solution of the boundary-layer equations offers decided advantages over others; while approximate, this analytical and rather general approach to the integration of the equations of motion and energy has proved to yield highly accurate results, such accuracy being assessed by direct comparison with the

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corresponding numerical solutions. Following up the basic work* [1, 2], Meksyn has successfully applied the procedure to determine the flow about an elliptic cylinder [3] and retarded incompressible flow past a semi-infinite plate [4]; a simple example of a compressible flow is discussed in [5]. An improvement of the method is contained in [6]. In his later papers [7, 8], Meksyn considered flows with variable physical properties demonstrating that (at least for simple types of property variations) the asymptotic method retains its accuracy there as well.

One purpose of this paper is to indicate the accuracy of the asymptotic method in solving systems of nonlinear differential equations in contrast, for example, to [1, 2] that deal with a single equation of the Falkner–Skan type. Partly for this reason and in order to have a ready comparison with corresponding numerical solutions, we consider a general three-dimensional flow of a viscous incompressible fluid in the vicinity of a stagnation point on a regular surface S. Numerical solutions providing the standards for accuracy are due to Howarth [10].

Recent interest in stagnation point heat transfer has provided the motivation for extending the work by considering the energy equation as well. The known solutions for plane and axisymetric flows are two special cases covered herein from a more general point of view.

II. ANALYSIS

1. *Basic equations and the associated boundary value problems*

Consider a steady three-dimensional laminar flow of an incompressible viscous fluid over a regular surface S. Dissipation effects are assumed to be negligible and all physical properties of the fluid are taken as constant. Under these conditions, the boundary layer equations expressing the principles of conservation of mass, linear momentum and energy in a body-oriented orthogonal coordinate frame take the form

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + h_1 h_2 \frac{\partial w}{\partial \zeta} = 0, \tag{2.1}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \nu\frac{\partial^2 u}{\partial \zeta^2}, \tag{2.2}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = -\frac{1}{\rho h_2}\frac{\partial p}{\partial \eta} + \nu\frac{\partial^2 v}{\partial \zeta^2}, \tag{2.3}$$

$$\frac{u}{h_1}\frac{\partial T}{\partial \xi} + \frac{v}{h_2}\frac{\partial T}{\partial \eta} + w\frac{\partial T}{\partial \zeta} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial \zeta^2}. \tag{2.4}$$

where ξ, η are coordinate curves on S and φ is measured in the direction of the local normal positive outward; the corresponding velocity components are u, v, w and h_1, h_2, h_3 are the usual coefficients related to a length element. T, p, ρ, ν and Pr denote respectively the temperature, pressure, density, kinematic viscosity and the Prandtl number.

Now, it may be easily shown [10] that in the vicinity of $P \equiv (0, 0, 0)$—a stagnation point on S—the system (2.1)–(2.4) may be replaced with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial \zeta} = 0, \tag{2.5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial \zeta} = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \nu\frac{\partial^2 u}{\partial \zeta^2}, \tag{2.6}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial \zeta} = U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \nu\frac{\partial^2 v}{\partial \zeta^2}, \tag{2.7}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial \zeta} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial \zeta^2}, \tag{2.8}$$

where the new coordinate axes Px, Py (obtained by rotating $P\xi$, $P\eta$) are so chosen that the

* The reader may also wish to consult Meksyn's book [9] containing a more complete bibliography of pertinent papers than that given herein.

irrotational mainstream has the components

$$U = ax, \qquad V = by, \tag{2.9}$$

a, b being constants. In other words, the flow near the stagnation point may be computed as if the surface were a plane, (x, y, ζ) forming a suitable Cartesian coordinate frame. Accordingly, we shall henceforth confine our attention to the system (2.5)–(2.8) subject to the following boundary conditions:

$$\left.\begin{aligned} &u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = 0, \\ &\qquad\qquad T(x, y, 0) = T_w = \text{const.} \\ &\lim_{\zeta\to\infty} u(x, y, \zeta) = U, \ \lim_{\zeta\to\infty} v(x, y, \zeta) = V, \\ &\quad \lim_{\zeta\to\infty} T(x, y, z) = T_\infty = \text{const.} \end{aligned}\right\} \tag{2.10}$$

We seek solutions where u, v are of the form

$$u = axf'(z), \qquad v = byg'(z); \tag{2.11}$$

$z \equiv (a/\nu)^{\frac{1}{2}}\zeta$ is a new independent variable and $' \equiv d/dz$.

Equation (2.5) and the boundary conditions (2.10) imply that a compatible representation for w is

$$w = -\left(\frac{\nu}{a}\right)^{\frac{1}{2}}(af + bg). \tag{2.12}$$

Introducing a new dependent variable

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{2.13}$$

and using (2.11), (2.12), it follows from (2.6)–(2.8) that

$$f''' + (f + cg)f''' = f'^2 - 1, \tag{2.14}$$

$$g''' + (f + cg)g'' = c(g'^2 - 1), \tag{2.15}$$

$$\theta'' + Pr(f + cg)\theta' = 0, \tag{2.16}$$

where $c \equiv b/a$ is a parameter. (Without loss of generality we may limit our consideration to the range $0 \leqslant c \leqslant 1$). Of course, (2.16) implies that $\theta = \theta(z)$ only.

The boundary conditions corresponding to (2.10) are:

$$\left.\begin{aligned} &f(0) = f'(o) = g(0) = g'(0) = 0, \\ &\qquad\qquad\qquad\qquad \theta(0) = 1; \\ &\lim_{z\to\infty} f'(z) = 1, \qquad \lim_{z\to\infty} g'(z) = 1, \\ &\qquad\qquad\qquad \lim_{z\to\infty} \theta(z) = 0. \end{aligned}\right\} \tag{2.17}$$

The system (2.14)–(2.17) has several interesting properties. We note first that for any given value of the parameter c, a solution (2.14)–(2.17) in fact satisfies the full Navier–Stokes equations expressed in a Cartesian coordinate system.* [To see this, observe that (2.14)–(2.17) remain unchanged when $\nu\,\nabla^2 u$, $\nu\,\nabla^2 v$ and $\nu/Pr\,\nabla^2 T$ replace respectively the last three terms on the right-hand sides of (2.6), (2.7) and (2.8).] When $c = 1$, clearly $f \equiv g$ and (2.14)–(2.17) give the solution for a stagnation point flow on a body of revolution.

If $c = b = 0$, we recover the classical two-dimensional flow. The situation is different if in (2.14)–(2.16) we take the limit as c approaches zero. The resultant equations,

$$f''' + ff'' = f'^2 - 1,$$

$$g''' + fg'' = 0,$$

$$\theta'' + Prf\theta' = 0,$$

are then identical to those governing the flow near the stagnation point (line) on a circular cylinder, unbounded in the y-direction with its axis inclined to the mainstream at angle $\alpha = \arctan\ V/U$. (Because our procedure is formal, there is no reason to expect that V is of the form (2.9). In fact V is constant.) Observe that the chordwise flow is unaffected by the spanwise motion (the g-flow). This situation typifies the so-called "independence principle" exploited for example in the papers

* We have an immediate generalization of the well known two-dimensional case. The potential flow is now $U = ax$, $V = by$, $W = -(a + b)z$ and the solutions of the equations of motion, being of the boundary-layer type, join this flow (up to the displacement effect) at infinity. The ζ-component of the momentum equation determines then the ζ-dependence of the pressure.

by Sears [11] and Görtler [12]. Finally, the temperature field is independent of the g-flow and formally identical to it when $Pr = 1$.

2. *Integration of the transformed equations of motion*

We turn now to the integration of the coupled nonlinear one parameter system (2.14), (2.15) subject to (2.17). The functions f, g are represented by the series

$$f(z) = \sum_{n=2}^{\infty} \frac{a_n}{n!} z^n, \qquad g(z) = \sum_{n=2}^{\infty} \frac{b_n}{n!} z^n, \tag{2.18}$$

the boundary conditions on f, g suggesting that both start with $n = 2$. Upon substituting (2.18) into (2.14), (2.15) and collecting powers of z we find the first few coefficients:

$$a_2 \equiv A, \quad a_3 = -1, \quad a_4 = 0, \quad a_5 = (A^2 - cAB), \ldots$$
$$b_2 \equiv B, \quad b_3 = -c, \quad b_4 = 0, \quad b_5 = -(AB - cB^2), \ldots$$

To obtain further terms it is convenient to use the recursive formulae

$$\begin{aligned} a_k = (k-3)! \Bigg[& -a_2 a_{k-3} \frac{(k-3)(k-4)}{2!(k-3)!} - a_3 a_{k-4} \frac{(k-4)(k-5)}{3!(k-4)!} - \ldots a_{k-3} a_2 \frac{1.2.1}{(k-3)!2!} \\ & - cb_2 a_{k-3} \frac{(k-3)(k-4)}{2!(k-3)!} - cb_3 a_{k-4} \frac{(k-4)(k-5)}{3!(k-4)!} - \ldots cb_{k-3} a_2 \frac{1.2.1}{(k-3)!2!} \\ & + a_2 a_{k-3} \frac{2(k-3)}{2!(k-3)!} + a_3 a_{k-4} \frac{3(k-4)}{3!(k-4)!} + \ldots a_{k-3} a_2 \frac{(k-3).2}{(k-3)!2!} \Bigg] \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} b_k = (k-3)! \Bigg[& -a_2 b_{k-3} \frac{(k-3)(k-4)}{2!(k-3)!} - a_3 b_{k-4} \frac{(k-4)(k-5)}{3!(k-4)!} - \ldots a_{k-3} b_2 \frac{1.2.1}{(k-3)!2!} \\ & - cb_2 b_{k-3} \frac{(k-3)(k-4)}{2!(k-3)!} - cb_3 b_{k-4} \frac{(k-4)(k-5)}{3!(k-4)!} - \ldots cb_{k-3} b_2 \frac{1.2.1}{(k-3)!2!} \\ & + cb_2 b_{k-3} \frac{2(k-3)!}{2!(k-3)!} + cb_3 b_{k-4} \frac{3(k-4)}{3!(k-4)!} + \ldots cb_{k-3} b_2 \frac{(k-3).2}{(k-3)!2!} \Bigg], \end{aligned}$$

valid for $k \geqslant 4$.

We consider for the moment (2.14), (2.15) formally as a linear system for f'', g'' and hence write

$$f''(z) = A\,e^{-F(z)} - e^{-F(z)} \int_0^z (1 - f'^2)\, e^{F(z')}\, dz', \tag{2.20}$$

$$g''(z) = B\,e^{-F(z)} - e^{-F(z)} \int_0^z c(1 - g'^2)\, e^{F(z')}\, dz', \tag{2.21}$$

where

$$F(z) \equiv \int_0^z (f + cg)\, dz'. \tag{2.22}$$

It follows now from (2.20), (2.21) that f'', g'' are of the form

$$f'' = e^{-F} \phi(z), \qquad g'' = e^{-F} \kappa(z), \tag{2.23}$$

where

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n, \qquad \kappa(z) = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} z^n \tag{2.24}$$

are "slowly varying" functions; the latter is implied by the asymptotic solutions of (2.14), (2.15). The coefficients α_n, β_n are obtained in a straightforward way from (2.23) and (2.18). To get α_n we write

$$\sum_{n=2}^{\infty} a_n \frac{n(n-1)}{n!} z^{n-2} = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n \tag{2.25}$$

$$\times \exp\left[-\sum_{n=2}^{\infty} (a_n + cb_n)\frac{1}{(n+1)!} z^{n+1}\right],$$

expand both sides and compare powers of z. The results are

$$\begin{aligned}
\alpha_0 &= a_2 \equiv A, \\
\alpha_1 &= a_3 = -1, \\
\alpha_2 &= a_4 = 0, \\
\alpha_3 &= a_5 + \alpha_0(a_2 + cb_2) = 2A^2, \\
\alpha_4 &= a_6 + \frac{4!}{3!}\alpha_1(a_2 + cb_2) + \alpha_6(a_3 + cb_3) = \\
&\qquad -(7A + cB),
\end{aligned} \tag{2.26}$$

where the last equalities follow from (2.19). Similarly, expressions for β_m are found to be

$$\begin{aligned}
\beta_0 &= b_2 \equiv B, \\
\beta_1 &= b_3 = -c, \\
\beta_2 &= b_4 = 0, \\
\beta_3 &= b_5 + \beta_0(a_2 + cb_2) = 2cB^2, \\
\beta_4 &= b_6 + \frac{4!}{3!}\beta_1(a_2 + cb_2) + \beta_0(a_3 + cb_3) = \\
&\qquad -(7Bc^2 + Ac).
\end{aligned} \tag{2.27}$$

The velocity field is deduced by integrating (2.23) which amounts to the evaluation of the integrals

$$f'(z) = \int_0^z e^{-F(z')}\phi(z')\,dz',$$

$$g'(z) = \int_0^z e^{-F(z')}\,\kappa(z')\,dz'. \tag{2.28}$$

This is done by the method of steepest descent. The integrals (2.28) are transformed as follows: we set

$$F(z) = z^3 \sum_{m=0}^{\infty} c_m z^m = \tau, \tag{2.29}$$

express z as a series in

$$z = \sum_{m=0}^{\infty} \frac{A_m}{m+1} \tau^{\frac{1}{3}(m+1)} \tag{2.30}$$

and hence obtain

$$f' = \int_0^{\tau} e^{-\tau'}\phi(z)\frac{dz}{d\tau'}d\tau',$$

$$g' = \int_0^{\tau} e^{-\tau'}\kappa(z)\frac{dz}{d\tau'}d\tau'. \tag{2.31}$$

With

$$\phi(z)\frac{dz}{d\tau} = \tau^{-\frac{2}{3}} \sum_{m=0}^{\infty} d_m\, \tau^{m/3},$$

there follow the desired forms of the integrals,

$$f' = \int_0^{\tau} e^{-\tau'}\tau'^{-\frac{2}{3}} \sum_{m=0}^{\infty} d_m\, \tau'^{m/3} d\tau',$$

$$g' = \int_0^{\tau} e^{-\tau'}\tau'^{-\frac{2}{3}} \sum_{m=0}^{\infty} e_m\, \tau'^{m/3} d\tau. \tag{2.32}$$

The latter are evaluated in terms of incomplete gamma functions; for each value of c, the quantities A, B are obtained from (2.32) and the conditions $\lim_{z\to\infty} f'(z) = 1$, $\lim_{z\to\infty} g'(z) = 1$, that is, simultaneous algebraic equations

$$1 = \sum_{m=0}^{\infty} d_m\, \Gamma[(m+1)/3],$$

$$1 = \sum_{m=0}^{\infty} e_m\, \Gamma[(m+1)/3], \tag{2.33}$$

for A, B with c as a parameter. Of course, $\{d_m\}$, $\{e_m\}$ must be expressed as functions of A, B and c. For what follows, it shall be convenient to deduce at this point the appropriate expressions for $\{c_m\}$, $\{A_m\}$ as well.

The c_m's are determined by

$$z^3 \sum_{m=0}^{\infty} c_m z^m = \int_0^z \sum_{n=0}^{\infty} \left(\frac{a_n}{n!} \tilde{z}_n + c \frac{b_n}{n!} \tilde{z}^n \right) dz,$$

an obvious consequence of (2.18), (2.22) and (2.29). Straightforward computations yield

$$c_0 = \frac{1}{3!}(a_2 + cb_2), \qquad c_1 = \frac{1}{4!}(a_3 + cb_3), \ldots,$$

$$c_m = \frac{1}{(m+3)!}(a_{m+2} + cb_{m+2}), \tag{2.34}$$

and, since a_m, b_m are known in terms of A, B, it follows that

$$\begin{aligned} c_0 &= \frac{1}{3!}(A + cB), \\ c_1 &= -\frac{1}{4!}(1 + c^2), \\ c_2 &= 0, \\ c_3 &= \frac{1}{6!}(A - cB)^2, \\ c_4 &= \frac{1}{7!}[A(4 - 6c + 4c^2 - 3c^3) \\ &\qquad + B(4c - 6c^2 + 4c^3)]. \end{aligned} \tag{2.35}$$

The A_m's are coefficients of z^m in the expression

$$(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)}.* \tag{2.36}$$

Therefore:

$$\begin{aligned} A_0 &= c_0^{-\frac{1}{3}} = [\tfrac{1}{6}(A + cB)]^{-\frac{1}{3}}, \\ A_1 &= -\tfrac{2}{3} c_1 c_0^{-\frac{5}{3}} = \tfrac{1}{9}(1 + c^2)[\tfrac{1}{6}(A + cB)]^{-\frac{5}{3}}, \\ A_2 &= c_1^2 c_0^{-3} = [\tfrac{1}{24}(1 + c^2)]^2 [\tfrac{1}{6}(A + cB)]^{-3}, \\ A_3 &= c_0^{-\frac{4}{3}} \left[-\frac{4}{3}\frac{c_3}{c_0} - \frac{140}{81}\left(\frac{c_1}{c_0}\right)^3 \right] = [\tfrac{1}{6}(A + cB)]^{-\frac{4}{3}}. \\ &\quad \cdot \left\{ -\frac{4}{3} \cdot \frac{1}{6!}(A - cB)^2 [\tfrac{1}{6}(A + cB)]^{-1} + \frac{140}{81} \cdot \frac{1}{4}\left(\frac{1 + c^2}{A + cB}\right)^3 \right\}, \end{aligned} \tag{2.37}$$

$$\begin{aligned} A_4 &= c_0^{-\frac{5}{3}} \left[-\frac{5}{3}\frac{c_4}{c_0} + \frac{40}{9}\frac{c_1 c_3}{c_0^2} + \frac{770}{243}\left(\frac{c_1}{c_0}\right)^4 \right] = [\tfrac{1}{6}(A + cB)]^{-\frac{5}{3}} \left\{ -\frac{5}{3} \cdot \frac{1}{7!}[A(4 - 6c + 4c^2 - 3c^3) \right. \\ &\quad \left. + B(4c - 6c^2 + 4c^3)][\tfrac{1}{6}(A + cB)]^{-1} - \frac{40}{9}\frac{1}{4!6!}\frac{(1 + c^2)(A - cB)^2}{[(1/3!)(A + cB)]^2} + \frac{770}{243}\left[\frac{3!}{4!}\frac{(1 + c^2)}{(A + cB)}\right]^4 \right\}, \\ A_5 &= \ldots. \end{aligned}$$

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* From (2.30) we have

$$dz = \sum_{m=0}^{\infty} \tfrac{1}{3} A_m \tau^{\frac{1}{3}(m-2)} d\tau$$

and hence

$$\int^{(0^+)} \frac{dz}{\tau^{\frac{1}{3}(m+1)}} = \tfrac{1}{3} A_m \int^{(0^+, 0^+, 0^+)} \frac{d\tau}{\tau} = 2\pi i A_m,$$

where (0^+) and $(0^+, 0^+. 0^+)$ denote respectively a positive single circuit around $z = 0$ and the corresponding triple circuit around $\tau = 0$. A_m is the coefficient of z^{-1} in the z expansion of $\tau^{-\frac{1}{3}(m+1)}$. But

$$\tau^{-\frac{1}{3}(m+1)} = z^{-(m+1)}(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)}$$

and hence A_m is the coefficient of z^m in $(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)}$.

It may be easily shown* that $3d_m$ and $3e_m$ are respectively the coefficients of z_m in the products

$$(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)} \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n$$

and

$$(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)} \sum_{n=0}^{\infty} \frac{\beta_n}{n!} z^n.$$

Hence, for $\alpha_2 = c_2 = 0$, we get

$$d_0 = \tfrac{1}{3} c_0^{-\frac{1}{3}},$$

$$d_1 = \frac{1}{3} c_0^{-\frac{2}{3}} \left(\alpha_1 - \frac{2}{3}\frac{c_1}{c_0}\alpha_0\right),$$

$$d_2 = \frac{1}{3} c_0^{-1} \left[-\frac{c_1}{c_0}\alpha_1 + \left(\frac{c_1}{c_0}\right)^2 \alpha_0\right],$$

$$d_3 = \frac{1}{3} c_0^{-\frac{4}{3}} \left\{\frac{\alpha_3}{3!} + \frac{14}{9}\left(\frac{c_1}{c_0}\right)^2 \alpha_1 - \left[\frac{4}{3}\frac{c_3}{c_0} + \frac{140}{81}\left(\frac{c_1}{c_0}\right)^3 \alpha_0\right]\right\},$$

$$d_4 = \frac{1}{3} c_0^{-\frac{5}{3}} \left\{\frac{\alpha_4}{4!} - \frac{5}{3}\frac{c_1}{c_0}\frac{\alpha_3}{3!} - \left[\frac{5}{3}\frac{c_3}{c_0} + \frac{220}{81}\left(\frac{c_1}{c_0}\right)^3\right]\alpha_1 + \left[-\frac{5}{3}\frac{c_4}{c_0} + \frac{40}{9}\frac{c_1 c_3}{c_0^2} + \frac{770}{243}\left(\frac{c_1}{c_0}\right)^4\right]\alpha_0\right\}, \tag{2.38}$$

$$d_5 = \ldots.$$

and together with (2.35)

$$d_0 = \tfrac{1}{3}[\tfrac{1}{6}(A + cB)]^{-\frac{1}{3}} A,$$

$$d_1 = \tfrac{1}{3}[\tfrac{1}{6}(A + cB)]^{-\frac{2}{3}} \left[-1 + \frac{A}{6}\frac{1 + c^2}{A + cB}\right],$$

$$d_2 = \tfrac{1}{3}[\tfrac{1}{6}(A + cB)]^{-1} \left[-\frac{1}{4}\frac{1 + c^2}{A + cB} + \left(\frac{1}{4}\frac{1 + c^2}{A + cB}\right)^2 A\right], \tag{2.39}$$

$$d_3 = \tfrac{1}{3}[\tfrac{1}{6}(A + cB)]^{-\frac{4}{3}} \left\{\frac{A^2}{3} + \frac{14}{9}\left[\frac{1}{16}\left(\frac{1 + c^2}{A + cB}\right)^2\right] - \left[\frac{4}{3}\left(\frac{1}{120}\frac{(A - cB)^2}{A + cB}\right) + \frac{140}{81}\left(-\frac{1}{4}\frac{(1 + c^2)}{A + cB}\right)^3\right]A\right\},$$

$$d_4 = \ldots.$$

* The procedure is analogous to that used to determine A_m.

The coefficients e_m, as functions of β_m may be obtained from (2.38) by replacing α_m with β_m. The final results, in terms of A, B and c, are the following:

$$\begin{aligned} e_0 &= \tfrac{1}{3}\left[\tfrac{1}{6}(A + cB)\right]^{-\frac{1}{3}}B, \\ e_1 &= \tfrac{1}{3}\left[\tfrac{1}{6}(A + cB)\right]^{-\frac{2}{3}}\left[-c + \frac{B}{6}\frac{1 + c^2}{A + cB}\right], \\ e_2 &= \tfrac{1}{3}\left[\tfrac{1}{6}(A + cB)\right]^{-1}\left[-\frac{c}{4}\left(\frac{1 + c^2}{A + cB}\right) + \left(\frac{1}{4}\frac{1 + c^2}{A + cB}\right)^2 B\right], \\ e_3 &= \tfrac{1}{3}\left[\tfrac{1}{6}(A + cB)\right]^{-\frac{4}{3}}\left\{\frac{cB^2}{3} + \frac{14}{9}\left[-\frac{c}{16}\left(\frac{1 + c^2}{A + cB}\right)^2\right]\right. \\ &\qquad \left. - \left[\frac{4}{3}\left(\frac{1}{120}\frac{(A - cB)^2}{A + cB}\right) + \frac{140}{81}\left(-\frac{1}{4}\frac{(1 + c^2)}{A + cB}\right)^3\right]B\right\}, \end{aligned} \tag{2.40}$$

and so on.

The basic quantities A and B, together specifying both the magnitude and direction of the resultant surface stress, are now obtained as solutions of the system

$$\begin{aligned} 1 &= d_0\Gamma(\tfrac{1}{2}) + d_1\Gamma(\tfrac{2}{3}) + d_2\Gamma(1) + d_3\Gamma(\tfrac{4}{3}) + \ldots, \\ 1 &= e_0\Gamma(\tfrac{1}{2}) + e_1\Gamma(\tfrac{2}{3}) + e_2\Gamma(1) + e_3\Gamma(\tfrac{4}{3}) + \ldots, \end{aligned} \tag{2.41}$$

where $\{d_m\}$, $\{e_m\}$, $m = 0, 1, \ldots$ are given by (2.39), (2.40). In summing the right-hand sides of (2.41), Euler's method is applied whenever necessary. Results based on five terms of (2.41) are in excellent agreement with the numerical solutions reported by Howarth [10]. A comparison is given in the Table 1.

The effect of repeated Euler transformations on the accuracy of A and B are indicated in Figs. 1 and 2. We observe in passing that the values of B near $c = 0$ are in better agreement with Howarth's results if no Euler transformation is used on the g-equation. This happens because when $c = 0$ the g-equation (2.15) is homogeneous and the resultant series converges rapidly. (As is well known, Euler's transformation does not necessarily improve the convergence of a convergent series.)

Table 1

	$c = 0$	$c = 0{\cdot}25$	$c = 0{\cdot}50$	$c = 0{\cdot}75$	$c = 1{\cdot}00$
$A = f''(o)$	1·227	1·245	1·265	1·289	1·315
$A = f''(o)$, Howarth	1·233	1·247	1·267	1·288	1·312
$B = g''(o)$	0·585	0·838	1·014	1·172	1·315
$B = g''(o)$, Howarth	0·570	0·805	0·998	1·164	1·312

3. *Heat transfer*

The integration of the energy equation (2.16) is straightforward. Clearly, this equation implies that

$$\theta' = \theta'_0 e^{-PrF}.$$

where

$$\theta'_0 \equiv \left.\frac{d\theta}{dz}\right|_{z=0}$$

and

$$F \equiv \int_0^z (f + cg)\,dz'.$$

Integrating once more and using the condition $\theta(0) = 1$ we obtain

$$\theta = 1 + \int_0^z \theta_0' \, e^{-PrF} \, dz'. \tag{2.42}$$

An equivalent expression in the τ variable is

$$\theta = 1 + \int_0^\tau \theta_0' \, e^{-Pr\tau'} . \tfrac{1}{3} \sum_{m=0}^{\infty} A_m \tau'^{\frac{1}{3}(m-2)} \, d\tau' \tag{2.43}$$

where the coefficients A_m are given by (2.37). The last integral may be evaluated in terms of incomplete gamma functions. Now, the essential quantity for heat-transfer calculations is θ_0'. The particular form,

$$\theta_0' = \frac{-Pr^{\frac{1}{3}}}{\frac{1}{3} \sum_{m=0}^{\infty} A_m Pr^{-m/3} \Gamma[(m+1)/3]} \tag{2.44}$$

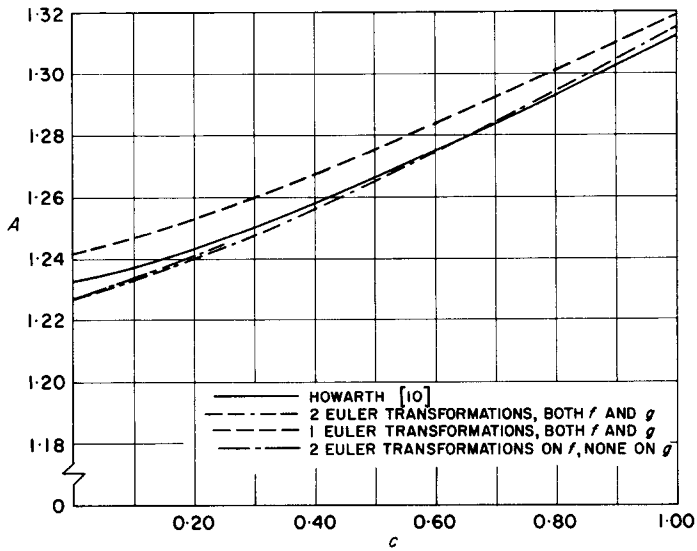

FIG. 1.

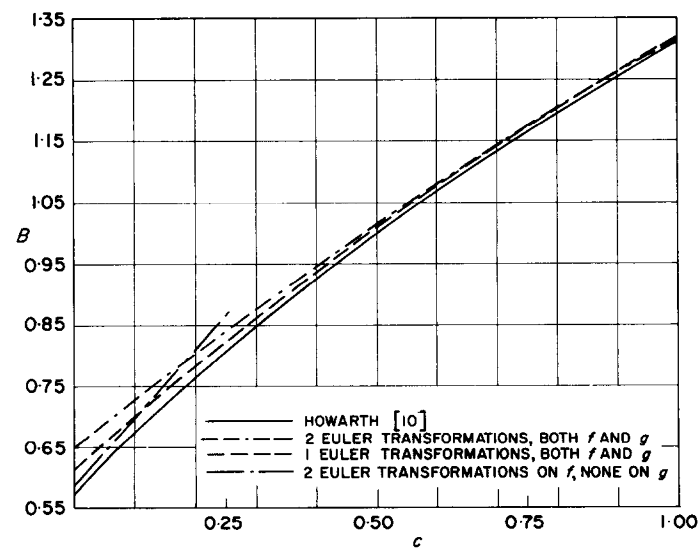

FIG. 2.

is deduced from (2.43) and the condition $\lim_{z\to\infty} \theta(z) = 0$. We prefer the equivalent expression

$$\theta_0' = \frac{-3\,\dfrac{1}{A_0\,\Gamma(\frac{1}{3})}\,Pr^{\frac{1}{3}}}{\displaystyle\sum_{m=0}^{\infty} \frac{A_m}{A_0}\,\frac{\Gamma[(m+1)/3]}{\Gamma(\frac{1}{3})}\,Pr^{-m/3}}, \qquad (2.45)$$

lending itself more readily to a comparison with the work of Chao and Jeng [13] in the special cases $c = 0, 1$. A few values of the coefficients $A_m/A_0 . \Gamma[(m+1)/3]/\Gamma(\frac{1}{3})$ and $3/A_0\,\Gamma(\frac{1}{3})$ are given respectively in Tables 2 and 3.

For $c = 0$ and 1, the slight differences between our coefficients in Tables 2 and 3, and those reported in [13] are attributed to the fact that we solve the equations of motion to obtain the values for $f''(0)$ whereas Chao and Jeng take the values from available numerical solutions. The influence on θ_0' is, however, quite small. For $Pr = 1$ we obtain for $c = 0, 0{\cdot}25, 0{\cdot}50, 0{\cdot}75$, and $1{\cdot}0$: $-\theta_0' = 0{\cdot}5718, 0{\cdot}6172, 0{\cdot}6639, 0{\cdot}7118, 0{\cdot}7589$, the variation being practically linear with c. Chao and Jeng give respectively, for $c = 0, 1$, $\theta_0' = -0{\cdot}5695$ and $\theta_0' = -0{\cdot}7637$; the numerical solution due to Sibulkin [14] for $c = 1$ is $\theta_0' = -0{\cdot}763$. Table 4 presents a comparison of our results and those of [14] for $c = 1$ and $Pr =$ 0·6, 1·0, 10. Figure 3 then indicates the variation of $-\theta_0'$ with c for variable Prandtl number.

Table 2

	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$c = 0$	1·0	0·11653	0·04464	0·001607	−0·00692
$c = 0$, Chao	1·0	0·11583	0·04410	0·00118	0·00020
$c = 0{\cdot}25$	1·0	0·09870	0·03209	0·003212	−0·00413
$c = 0{\cdot}50$	1·0	0·08974	0·02618	0·006633	−0·00064
$c = 0{\cdot}75$	1·0	0·08524	0·02389	0·00853	0·00178
$c = 1{\cdot}00$	1·0	0·084581	0·02338	0·00903	0·00318
$c = 1{\cdot}00$, Chao	1·0	0·08460	0·02352	0·00911	0·00021

Table 3

$c = 0$	$c = 0{\cdot}25$	$c = 0{\cdot}50$	$c = 0{\cdot}75$	$c = 1{\cdot}00$
0·6609	0·6978	0·7448	0·7968	0·8501
0·6608, Chao				0·8500, Chao

Table 4

Pr	0·6	1·0	10·0
$-\theta_0'$, [14]	0·625	0·763	1·76
$-\theta_0'$, [equation (2.45)]	0·621	0·759	1·75

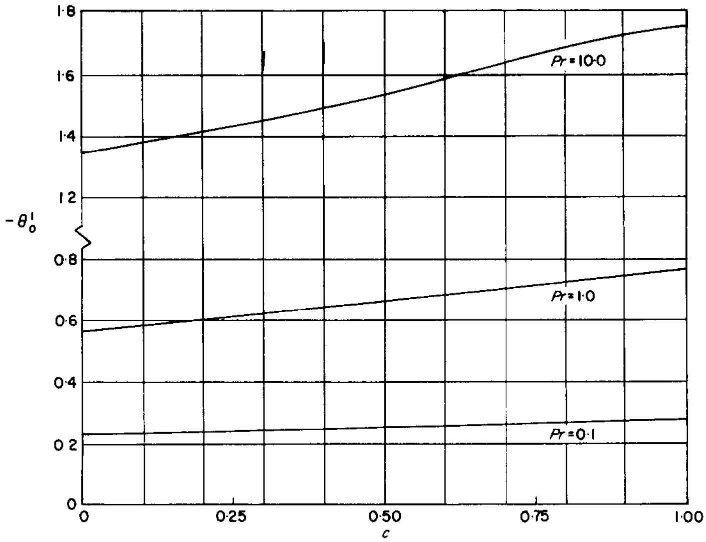

FIG. 3.

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Résumé—Les équations du mouvement et de l'énergie qui régissent l'écoulement général tri-dimensionnel d'un fluide incompressible près d'un point d'arrêt sont intégrées analytiquement. Le frottement local et le transport de chaleur sont déterminés lorsque la surface est isotherme. Chaque fois que cela est possible, ces résultats sont comparés avec les solutions numériques disponibles et l'on a trouvé qu'ils sont extrêmement précis. Ils montrent que la méthode asymptotique de résolution des équations de la couche limite garde sa précision lorsqu'on l'applique à un système d'équations non linéaires couplées, en faisont bien attention de sommer les séries divergentes par la méthode d'Euler.

Zusammenfassung—Die Bewegungs- und Energiegleichungen, welche die allgemeine dreidimensionale Strömung eines inkrompressiblen Mediums nahe dem Staupunkt kennzeichnen werden analytisch integriert. Oberflächenreibung und Wärmeübergang werden örtlich bestimmt bei isothermer Oberfläche. Soweit möglich werden diese Ergebnisse mit verfügbaren numerischen Lösungen verglichen, wobei sich ihre hohe Genauigkeit zeigt. Die asymptotische Lösungsmethode der Grenzschichtgleichungen behält ihre Genauigkeit bei, wenn sie auf ein System gekoppelter nichtlinearer Gleichungen angewandt und, wenn der Summierung divergenter Reihen nach der Euler-Methode besondere Sorgfalt gewidmet wird.

Аннотация—Проводится аналитическое интегрирование уравнений движения и энергии для трехмерного течения несжимаемой жидкости вблизи критической точки. Для изотермической поверхности найдены локальные коэффициенты теплообмена и трения. Полученные результаты сравниваются с известными численными решениями, причем соответствие оказалось хорошим. Установлено, что асимптотический метод решения уравнений пограничного слоя дает точные результаты для системы двойных нелинейных уравнений пограничного слоя, однако следует тщательно проводить суммирование расходящегося ряда методом Эйлера.